# Vascular Fluid Structure Simulation

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## Abstract

The vascular system is an important component for the human health and a computational model of blood flow could help diagnosis and treatment of health problems. Also, this project evaluates the stability of the solver to handle fluid structure interaction problem with the boundary implementation. Blood flow is described by 3D cylindrical incompressible Navier-Stokes equations (INS), and a set of structure equations determines the radial and longitudinal deformation of the vessel wall. Parallel Interoperable Computational Mechanics System Simulator (PICMSS) is chosen to solve INS. PICMSS is a parallel computational software for solving equations with continuous Galerkin finite element method and is written in C program with MPI and uses Trilinos iterative library for solving systems of linear equations generated internally by finite element method. On the other hand, I use continuous Galerkin finite element method and Newmark method to solve the structure equations.

## 1 Overview

This report is to simulate vascular flow in arteries by using incompressible Navier-Stokes equations(INS),which describe blood velocity and pressure, and a set of structure equations that determines the radial and longitudinal deformation of the vessel wall.

The main goal is to evaluate the stability of implemented solvers to handle fluid structure interaction problems. The fluid-structure equations are solved by continuous Galerkin finite element method and will extend to discontinuous Galerkin finite element method. This project also utilizes DIEL to solve weak coupling equations.



To solve the equations, Parallel Interoperable Computational Mechanics System Simulator(PICMSS) was chosen. PICMSS is a parallel computational software for solving equations with continuous Galerkin finite element method.

INS is solved by continuous Galerkin finite element method with the initial conditions and boundary solutions from Quarteroni et al. [1]. For 1D and 2D axisymmetric structure equations, I first implemented the algorithm presented in Ottesen et al. [2], then use continuous Galerkin finite element method and also Newmark method in Hughes [3].For 3D structure equations, I use the approach from Raoul et al. [4], then use continuous Galerkin finite element method.

## 2 Fluid-Structure Interactions

There are two major components in fluid-structure interactions, fluid(blood) and solid structure(vessel wall).They affect each other. Blood flow causes deformation of the vessel wall and deformation of the wall changes the boundary conditions of blood flow.
Fluid (blood) is modeled by Navier-Stokes equations. Solid structure (vessel wall) is modeled by partial differential equations of 1D, 2D and 3D, giving radial and longitudinal deformation of wall from its resting state. This project develop a coupling strategy to solve fluid-structure equations.

## 2.1 Fluid Equations

$$\begin{cases} \mathbf{u}_t - \frac{1}{Re}\nabla^2\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = 0 \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

## 2.2 Structure Equations

$$\begin{cases} \nabla\cdot\tau^s - \nabla\cdot p^s = 0 \\ det(\mathbf{F}) = 0 \end{cases}$$

$$where \qquad \tau^s = G(\mathbf{F}\cdot\mathbf{F}^T - \mathbf{I}),$$
$$\mathbf{F} = (\vec{\nabla}_0\vec{x})^T$$

## 2.3 Algorithm

1. Solve Navier-Stokes equations(INS) for blood flow velocity and pressure
2. Solve structure equations for deformations of the vessel wall
3. Update the mesh and radial velocity at vessel wall
4. Repeat Step 1-3 until a stable solution is reached
5. t = t + Δt
6. Continue from Step 1

# 3 2D Axisymmetric Fluid equations

I assume that blood flow is axisymmetric and without swirl. Therefore, the fluid equations are derived using cylindrical representation$(r,x,\theta)$ of the incompressible Navier-Stokes equations with no $\theta$ component, where $x$ is in axial direction, $r$ is in radial direction and $\theta$ is angular coordinate.Hence, the fluid equations take the form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial x^2}-\frac{u}{r^2}) \quad (1)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial x^2}) \quad (2)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial x} = 0, \quad (3)$$

where $u$ is the radial velocity, $w$ is the longitudinal velocity, $p$ is blood pressure, $\rho$ is the density of blood (constant, $1g/cm^3$), and $\nu=\mu/\rho$ is the kinematic viscosity (also constant, $0.035cm/s$).

## 3.1 Mathematical transformation of Fluid Equation

The fluid equations are reduced to a matrix form through transformation to weak finite element form and semi-discretization.

Semi-discretization:

$$u(x,r)\approx u^e(x,r)=\sum_{j=1}^{n}U_j^e\psi_j^e(x,r)$$

$$w(x,r)\approx w^e(x,r)=\sum_{j=1}^{n}W_j^e\psi_j^e(x,r)$$

$$p(x,r)\approx p^e(x,r)=\sum_{j=1}^{n}P_j^e\psi_j^e(x,r)$$

$$1/r=\sum_{j=1}^{n}\{IR\}_j^e\psi_j^e(x,r)$$

$$1/r^2=\sum_{j=1}^{n}\{IR2\}_j^e\psi_j^e(x,r)$$

### 3.1.1 Fluid Equation (1)

$$\begin{aligned}
0 &= \int_{\Omega^e}\psi_i^e\{\frac{\partial\sum_{j=1}^n U_j^e\psi_j^e}{\partial t}+(\sum_{k=1}^{n}U_k^e\psi_k^e)(\frac{\partial\sum_{j=1}^n U_j^e\psi_j^e}{\partial r}) \\
&+ (\sum_{k=1}^{n}W_k^e\psi_k^e)(\frac{\partial\sum_{j=1}^n U_j^e\psi_j^e}{\partial x}) \\
&+ \frac{1}{\rho}\frac{\partial\sum_{j=1}^n P_j^e\psi_j^e}{\partial r}-\nu(\frac{1}{r}\frac{\partial\sum_{j=1}^n U_j^e\psi_j^e}{\partial r}-\frac{\sum_{j=1}^n U_j^e\psi_j^e}{r^2})\}dxdr \\
&+ \int_{\Omega^e}\nu(\frac{\partial\psi_i^e}{\partial x}\frac{\partial\sum_{j=1}^n U_j^e\psi_j^e}{\partial x}+\frac{\partial\psi_i^e}{\partial r}\frac{\partial\sum_{j=1}^n U_j^e\psi_j^e}{\partial r})dxdr \\
&- \oint_{\Gamma^e}\nu\psi_i^e(\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial r}n_r)ds \\
0 &= \sum_{j=1}^{n}\int_{\Omega^e}\psi_i^e\psi_j^e dxdr\frac{\partial U_j^e}{\partial t}+\sum_{j=1}^{n}\sum_{k=1}^{n}U_k^e\int_{\Omega^e}\psi_k^e\psi_i^e\frac{\partial\psi_j^e}{\partial r}dxdrU_j^e \\
&+ \sum_{j=1}^{n}\sum_{k=1}^{n}W_k^e\int_{\Omega^e}\psi_k^e\psi_i^e\frac{\partial\psi_j^e}{\partial x}dxdrU_j^e \\
&+ \sum_{j=1}^{n}\int_{\Omega^e}\frac{1}{\rho}\psi_i^e\frac{\partial\psi_j^e}{\partial r}dxdrP_j^e \\
&+ \sum_{j=1}^{n}\int_{\Omega^e}\nu\{\frac{\psi_i^e\psi_j^e}{r^2}+\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}+(\frac{\partial\psi_i^e}{\partial r}-\frac{\psi_i^e}{r})\frac{\partial\psi_j^e}{\partial r}\}dxdrU_j^e \\
&- \oint_{\Gamma^e}\nu\psi_i^e(\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial r}n_r)ds
\end{aligned}$$

$$\begin{aligned}
M_{ij}^e &= \int_{\Omega^e}\psi_i^e\psi_j^e dxdr \\
&= DET_e[B200]
\end{aligned}$$

$$\begin{aligned}
A_{ijk}^e &= \int_{\Omega^e}\psi_k^e\psi_i^e\frac{\partial\psi_j^e}{\partial r}dxdr \\
&= DET_e[B300y]
\end{aligned}$$

$$\begin{aligned}
B_{ijk}^e &= \int_{\Omega^e}\psi_k^e\psi_i^e\frac{\partial\psi_j^e}{\partial x}dxdr \\
&= DET_e[B300x]
\end{aligned}$$

$$\begin{aligned}
C_{ij}^e &= \int_{\Omega^e}\frac{1}{\rho}\psi_i^e\frac{\partial\psi_j^e}{\partial r}dxdr \\
&= \frac{1}{\rho}DET_e[B20y]
\end{aligned}$$

$$\begin{aligned}
K_{ij}^e &= \int_{\Omega^e}\nu\{\frac{\psi_i^e\psi_j^e}{r^2}+\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}+(\frac{\partial\psi_i^e}{\partial r}-\frac{\psi_i^e}{r})\frac{\partial\psi_j^e}{\partial r}\}dxdr \\
&= \nu\int_{\Omega^e}(\sum_{k=1}^{n}\{IR2\}_k^e\psi_k^e)\psi_i^e\psi_j^e \\
&- (\sum_{k=1}^{n}\{IR\}_k^e\psi_k^e)\psi_i^e\psi_j^e+\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}+\frac{\partial\psi_i^e}{\partial r}\frac{\partial\psi_j^e}{\partial r}dxdr \\
&= \nu DET_e(\{IR2\}^T[B3000]-\{IR\}^T[B3000]+[B2kk])
\end{aligned}$$

$$[M_1^e]\{DU^e\}+\{U^e\}^T[A_1^e]\{U^e\}+\{W^e\}^T[B_1^e]\{U^e\}$$
$$+[C_1^e]\{P^e\}+[K_1^e]\{U^e\}=0$$

### 3.1.2 Fluid Equation (2)

$$\begin{aligned}
0 &= \int_{\Omega^e}\psi_i^e\{\frac{\partial\sum_{j=1}^n W_j^e\psi_j^e}{\partial t}+(\sum_{k=1}^n U_k^e\psi_k^e)(\frac{\partial\sum_{j=1}^n W_j^e\psi_j^e}{\partial r})\\
&+ (\sum_{k=1}^n W_k^e\psi_k^e)(\frac{\partial\sum_{j=1}^n W_j^e\psi_j^e}{\partial x})\\
&+ \frac{1}{\rho}\frac{\partial\sum_{j=1}^n P_j^e\psi_j^e}{\partial x}-\nu(\frac{1}{r}\frac{\partial\sum_{j=1}^n W_j^e\psi_j^e}{\partial r})dxdr\\
&+ \int_{\Omega^e}\nu(\frac{\partial\psi_i^e}{\partial x}\frac{\partial\sum_{j=1}^n W_j^e\psi_j^e}{\partial x}+\frac{\partial\psi_i^e}{\partial r}\frac{\partial\sum_{j=1}^n W_j^e\psi_j^e}{\partial r})dxdr\\
&- \oint_{\Gamma^e}\nu\psi_i^e(\frac{\partial w}{\partial x}n_x+\frac{\partial w}{\partial r}n_r)ds\\
0 &= \sum_{j=1}^n\int_{\Omega^e}\psi_i^e\psi_j^e dxdr\frac{\partial W_j^e}{\partial t}+\sum_{j=1}^n\sum_{k=1}^n U_k^e\int_{\Omega^e}\psi_k^e\psi_i^e\frac{\partial\psi_j^e}{\partial r}dxdrW_j^e\\
&+ \sum_{j=1}^n\sum_{k=1}^n W_k^e\int_{\Omega^e}\psi_k^e\psi_i^e\frac{\partial\psi_j^e}{\partial x}dxdrW_j^e\\
&+ \sum_{j=1}^n\int_{\Omega^e}\frac{1}{\rho}\psi_i^e\frac{\partial\psi_j^e}{\partial x}dxdrP_j^e\\
&+ \sum_{j=1}^n\int_{\Omega^e}\nu\{\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}+(\frac{\partial\psi_i^e}{\partial r}-\frac{\psi_i^e}{r})\frac{\partial\psi_j^e}{\partial r}\}dxdrW_j^e\\
&- \oint_{\Gamma^e}\nu\psi_i^e(\frac{\partial w}{\partial x}n_x+\frac{\partial w}{\partial r}n_r)ds\\
M_{ij}^e &= \int_{\Omega^e}\psi_i^e\psi_j^e dxdr\\
&= DET_e[B200]
\end{aligned}$$

$$\begin{aligned}
A_{ijk}^e &= \int_{\Omega^e}\psi_k^e\psi_i^e\frac{\partial\psi_j^e}{\partial r}dxdr\\
&= DET_e[B300y]
\end{aligned}$$

$$\begin{aligned}
B_{ijk}^e &= \int_{\Omega^e}\psi_k^e\psi_i^e\frac{\partial\psi_j^e}{\partial x}dxdr\\
&= DET_e[B300x]
\end{aligned}$$

$$\begin{aligned}
C_{ij}^e &= \int_{\Omega^e}\frac{1}{\rho}\psi_i^e\frac{\partial\psi_j^e}{\partial x}dxdr\\
&= \frac{1}{\rho}DET_e[B20x]
\end{aligned}$$

$$\begin{aligned}
K_{ij}^e &= \int_{\Omega^e}\nu\{\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}+(\frac{\partial\psi_i^e}{\partial r}-\frac{\psi_i^e}{r})\frac{\partial\psi_j^e}{\partial r}\}dxdr\\
&= \nu\int_{\Omega^e}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}+\frac{\partial\psi_i^e}{\partial r}\frac{\partial\psi_j^e}{\partial r}-(\sum_{k=1}^n\{IR\}_k^e\psi_k^e)\psi_i^e\psi_j^e dxdr\\
&= \nu DET_e([B2kk]-\{IR\}^T[B3000])
\end{aligned}$$

$$[M_2^e]\{DW^e\}+\{U^e\}^T[A_2^e]\{W^e\}+\{W^e\}^T[B_2^e]\{W^e\}+[C_2^e]\{P^e\}+[K_2^e]\{W^e\}=0$$

## 3.2 Projection Method

The first step is using Euler backward method to approximate $\frac{\partial u}{\partial t}, \frac{\partial w}{\partial t}$. Then, the pressure(p) is replaced by SPHI, which is corrected by PHI for each step. PHI satisfies the following equation:

$$\nabla^2 PHI=\frac{u}{r}+\frac{\partial u}{\partial r}+\frac{\partial w}{\partial x}$$

Next, CG method is applied to above equation:

$$0=\sum_{j=1}^n\int_{\Omega_e}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}+\frac{\partial\psi_i^e}{\partial r}\frac{\partial\psi_j^e}{\partial r}dxdrPHI_j^e+\sum_{j=1}^n\int_{\Omega_e}\frac{\psi_i^e\psi_j^e}{r}+\psi_i^e\frac{\partial\psi_j^e}{\partial r}dxdrU_j^e$$
$$+\sum_{j=1}^n\int_{\Omega_e}\psi_i^e\frac{\partial\psi_j^e}{\partial x}dxdrW_j^e$$

$$[D^e]\{PHI^e\}+[E^e]\{U^e\}+[F^e]\{W^e\}=0$$

$$\begin{aligned}
D_{ij}^e &= \int_{\Omega_e}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}+\frac{\partial\psi_i^e}{\partial r}\frac{\partial\psi_j^e}{\partial r}dxdr\\
&= DET_e[B2kk]\\
E_{ij}^e &= \int_{\Omega_e}\frac{\psi_i^e\psi_j^e}{r}+\psi_i^e\frac{\partial\psi_j^e}{\partial r}dxdr\\
&= \sum_{k=1}^n\{IR\}_k^e\int_{\Omega_e}\psi_k^e\psi_i^e\psi_j^e dxdr+\int_{\Omega_e}\psi_i^e\frac{\partial\psi_j^e}{\partial r}dxdr\\
&= DET_e(\{IR\}^T[B3000]+[B20y])\\
F_{ij}^e &= \int_{\Omega_e}\psi_i^e\frac{\partial\psi_j^e}{\partial x}dxdr=DET_e[B20x]
\end{aligned}$$

Hence, the whole system is as follow and it is solved by PICMSS:

$$[M_1^e]\{\frac{U^k-U^{k-1}}{\delta t}\}+\{U^k\}^T[A_1^e]\{U^k\}+\{W^k\}^T[B_1^e]\{U^k\}$$
$$+[C_1^e]\{SPHI^{k-1}\}+[K_1^e]\{U^k\}=0$$
$$[M_2^e]\{\frac{W^k-W^{k-1}}{\delta t}\}+\{U^k\}^T[A_2^e]\{W^k\}+\{W^k\}^T[B_2^e]\{W^k\}$$
$$+[C_2^e]\{SPHI^{k-1}\}+[K_2^e]\{W^k\}=0$$
$$[D^e]\{PHI^e\}+[E^e]\{U^e\}+[F^e]\{W^e\}=0$$
$$SPHI^k=SPHI^{k-1}+PHI$$

The PICMSS code and result are present in the Appendix.

# 4 Structure Equations

Structure equations are based on the Ottesen's formula[2].

$$M_0\frac{\partial^2\xi}{\partial t^2}+L_x\frac{\partial\xi}{\partial t}+K_x\xi=\frac{E_xh}{1-\sigma_\theta\sigma_x}\frac{\partial^2\xi}{\partial x^2}+(\frac{E_xh\sigma_x}{a(1-\sigma_\theta\sigma_x)}+\frac{T_{t_0}-T_{\theta_0}}{a})\frac{\partial\eta}{\partial x}$$

$$-\mu[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial x}]_a$$

$$M_0\frac{\partial^2\eta}{\partial t^2}+L_r\frac{\partial\eta}{\partial t}+K_r\eta=(-\frac{E_\theta h}{a^2(1-\sigma_\theta\sigma_x)}+\frac{T_{\theta_0}}{a^2})\eta+T_{t_0}\frac{\partial^2\eta}{\partial x^2}$$

$$-\frac{E_\theta h\sigma_\theta}{a(1-\sigma_\theta\sigma_x)}\frac{\partial\xi}{\partial x}+[p-2\mu\frac{\partial u}{\partial r}]_a$$

where $\xi$,$\eta$ represent longitudinal and radial deformations of the vessel wall respectively.

## 4.1 Physical constants

Parameters are

- $h$ thickness of wall
- $a \approx 10^{-3} m$ radius of artery
- $E_x$, $E_\theta$ Young's modulus in the $x$ and $\theta$ directions. $E_x/E_\theta \approx 1.2$.
- $M_a$, $L_x, L_r \approx 17 \times 10^3 kg/(sm^2)$, $K_x, K_r \approx 33 \times 10^3 kg/(s^2m^2)$ are the coefficients from modeling the tethering force as a dash pot. $M_a$ is the additional mass of the dash pot system, $L_r$ and $L_x$ are the frictional coefficients, and $K_r, K_x$ are the spring coefficients.
- $M_0 = M_a + \rho_0 h \approx 4kg/m^2$ where $\rho_0$ is the density of the wall
- $T_{t_0}$,$T_{\theta_0} \approx 0$ reference state of stresses in the longitudinal and circumferential directions
- $\sigma_x$=$\sigma_\theta = 0.29$ Poisson ration in the $x$ and $\theta$ directions
- $\nu = \mu/\rho$ kinematic viscosity
- $\rho \approx 10^3 kg/m^3$ density of blood
- $c_0 = E_\theta h/(2a\rho) \approx 5m/s$ Moens-Korteweg wave propagation factor

## 4.2 Mathematical transformation of First structural equation

$$M_0\frac{\partial^2\xi}{\partial t^2}+L_x\frac{\partial\xi}{\partial t}+K_x\xi=\frac{E_xh}{1-\sigma_\theta\sigma_x}\frac{\partial^2\xi}{\partial x^2}+(\frac{E_xh\sigma_x}{a(1-\sigma_\theta\sigma_x)}+\frac{T_{t_0}-T_{\theta_0}}{a})\frac{\partial\eta}{\partial x}$$

$$-\mu[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial x}]_a$$

Strong form:

$$\begin{aligned}0 &= \int_{\Omega^e}\psi_i^e\{M_0\frac{\partial^2\xi}{\partial t^2}+L_x\frac{\partial\xi}{\partial t}+K_x\xi-\frac{E_xh}{1-\sigma_\theta\sigma_x}\frac{\partial^2\xi}{\partial x^2}\\ &- (\frac{E_xh\sigma_x}{a(1-\sigma_\theta\sigma_x)}+\frac{T_{t_0}-T_{\theta_0}}{a})\frac{\partial\eta}{\partial x}\\ &- \mu[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial x}]_a\}dxdr\end{aligned}$$

Plugging in the combined identities and divergence theorem:

$$\begin{aligned}0 &= \int_{\Omega^e}\{\psi_i^eM_0\frac{\partial^2\xi}{\partial t^2}+\psi_i^eL_x\frac{\partial\xi}{\partial t}+\psi_i^eK_x\xi\}dxdr\\ &- \int_{\Omega^e}\{\frac{E_xh}{1-\sigma_\theta\sigma_x}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\xi}{\partial x}-\psi_i^e(\frac{E_xh\sigma_x}{a(1-\sigma_\theta\sigma_x)}+\frac{T_{t_0}-T_{\theta_0}}{a})\frac{\partial\eta}{\partial x}\\ &- \psi_i^e\mu[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial x}]_a\}dxdr\\ &- \oint_{\Gamma^e}\psi_i^e\frac{\partial\xi}{\partial x}n_xds\end{aligned}$$

where $n_x$ is the component of the unit normal vector

$$\hat{n}=n_x\hat{x}+n_r\hat{r}$$

on the boundary $\Gamma^e$ and $ds$ is the arclength of an infinitesimal line element along the boundary. Let

$$q_n=\frac{\partial\xi}{\partial x}n_x$$

Semi-discretization:

$$\xi(x,r)\approx\xi^e(x,r)=\sum_{j=1}^{n}X_j^e\psi_j^e(x,r)$$

$$w(x,r)\approx\sum_{j=1}^{n}W_j^e\psi_j(x,r)$$

$$\begin{aligned}0 &= \int_{\Omega^e}\{\psi_i^eM_0\frac{\partial^2\sum_{j=1}^nX_j^e\psi_j^e}{\partial t^2}+\psi_i^eL_x\frac{\partial\sum_{j=1}^nX_j^e\psi_j^e}{\partial t}+\psi_i^eK_x\sum_{j=1}^{n}X_j^e\psi_j^e\}dxdr\\ &- \int_{\Omega^e}\{\frac{E_xh}{1-\sigma_\theta\sigma_x}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\sum_{j=1}^nX_j^e\psi_j^e}{\partial x}-\psi_i^e(\frac{E_xh\sigma_x}{a(1-\sigma_\theta\sigma_x)}+\frac{T_{t_0}-T_{\theta_0}}{a})\frac{\partial\sum_{j=1}^nN}{\partial x}\\ &- \int_{\Omega^e}\psi_i^e\mu(\frac{\partial\sum\{W_j\}\psi_j}{\partial r}+\frac{\partial\sum\{U_j\}\psi_j}{\partial x}dxdr)-\oint_{\Gamma^e}\psi_i^eq_nds\end{aligned}$$

$$\begin{aligned}0 &= \sum_{j=1}^{n}\int_{\Omega^e}\{\psi_i^eM_0\psi_j^e\frac{\partial^2X_j^e}{\partial t^2}+\psi_i^eL_x\psi_j^e\frac{\partial X_j^e}{\partial t}+\psi_i^e\psi_j^eK_xX_j^e\\ &- \frac{E_xh}{1-\sigma_\theta\sigma_x}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}X_j^e\}dxdr\\ &- \sum_{j=1}^{n}\int_{\Omega^e}\psi_i^e(\frac{E_xh\sigma_x}{a(1-\sigma_\theta\sigma_x)}+\frac{T_{t_0}-T_{\theta_0}}{a})\frac{\partial\psi_j^e}{\partial x}N_j^edxdr\\ &- \mu\sum\int\psi_i\frac{\partial\psi_j}{\partial r}dxdr\{W_j\}-\mu\sum\int\psi_i\frac{\partial\psi_j}{\partial x}dxdr\{U_j\}-\oint_{\Gamma^e}\psi_i^eq_nds\end{aligned}$$

$$
\begin{aligned}
0 &= \sum_{j=1}^{n}\int_{\Omega^e}\{\psi_i^e M_0\psi_j^e\frac{\partial^2 X_j^e}{\partial t^2}+\psi_i^e L_x\psi_j^e\frac{\partial X_j^e}{\partial t} \\
&+ (\psi_i^e\psi_j^e K_x-\frac{E_xh}{1-\sigma_\theta\sigma_x}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x})X_j^e\}dxdr \\
&- \sum_{j=1}^{n}\int_{\Omega^e}\psi_i^e(\frac{E_xh\sigma_x}{a(1-\sigma_\theta\sigma_x)}+\frac{T_{t_0}-T_{\theta_0}}{a})\frac{\partial\psi_j^e}{\partial x}N_j^e dxdr \\
&- \mu\sum\int\psi_i\frac{\partial\psi_j}{\partial r}dxdr\{W_j\}-\mu\sum\int\psi_i\frac{\partial\psi_j}{\partial x}dxdr\{U_j\} \\
&- \oint_{\Gamma^e}\psi_i^e q_n ds \\
0 &= \sum_{j=1}^{n}\int_{\Omega^e}\psi_i^e M_0\psi_j^e dxdr\frac{\partial^2 X_j^e}{\partial t^2}+\psi_i^e L_x\psi_j^e dxdr\frac{\partial X_j^e}{\partial t} \\
&+ \sum_{j=1}^{n}\int_{\Omega^e}(\psi_i^e\psi_j^e K_x-\frac{E_xh}{1-\sigma_\theta\sigma_x}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x})dxdrX_j^e \\
&- \sum_{j=1}^{n}\int_{\Omega^e}\psi_i^e(\frac{E_xh\sigma_x}{a(1-\sigma_\theta\sigma_x)}+\frac{T_{t_0}-T_{\theta_0}}{a})\frac{\partial\psi_j^e}{\partial x}dxdrN_j^e \\
&- \mu\sum\int\psi_i\frac{\partial\psi_j}{\partial r}dxdr\{W_j\}-\mu\sum\int\psi_i\frac{\partial\psi_j}{\partial x}dxdr\{U_j\} \\
&- \oint_{\Gamma^e}\psi_i^e q_n ds
\end{aligned}
$$

### 4.2.1 1D version

$$
\begin{aligned}
DET_e &= ||\alpha'(s)|| \\
J_1 &= \frac{\partial s}{\partial x} \\
M_{ij}^e &= \int_{\Omega^e}M_0\psi_i^e\psi_j^e dxdr \\
&= 4\times DET_e[A200] \\
C_{ij}^e &= \int_{\Omega^e}L_x\psi_i^e\psi_j^e dxdr \\
&= 17\times10^3\times DET_e[A200] \\
K_{ij}^e &= \int_{\Omega^e}K_x\psi_i^e\psi_j^e-\frac{E_xh}{1-\sigma_\theta\sigma_x}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}dxdr \\
&= \int_0^1 K_x\psi_i^e\psi_j^e||\alpha'(s)||-\frac{E_xh}{1-\sigma_\theta\sigma_x}\frac{d\psi_i^e}{ds}\frac{d\psi_j^e}{ds}(\frac{\partial s}{\partial x})^2||\alpha'(s)||ds \\
&= 33\times10^3\times DET_e[A200]-13.1\times(J_1)^2DET_e[A2xx] \\
D_i^e &= \int_{\Omega^e}\psi_i^e(\frac{E_xh\sigma_x}{a(1-\sigma_\theta\sigma_x)}+\frac{T_{t_0}-T_{\theta_0}}{a})\frac{\partial\psi_j^e}{\partial x}dxdr \\
&= 3.8\times10^3\times DET_eJ_1[A20x] \\
S_i^e &= \int_{\Omega^e}\psi_i^e\mu[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial x}]_a dxdr \\
&= 3.5\times10^{-3}\times[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial x}]_a\times DET_e[A10] \\
Q_i^e &= \oint_{\Gamma^e}\psi_i^e q_n ds = 0
\end{aligned}
$$

$$
\{M_1^e\}\frac{\partial^2}{\partial t^2}\{X^e\}+\{C_1^e\}\frac{\partial}{\partial t}\{X^e\}+\{K_1^e\}\{X^e\}+\{D_1^e\}\{N^e\}
$$
$$
=\{Q_1^e\}+\{S_1^e\}
$$

### 4.2.2 Newmark Method

Apply Newmark method to the structure equations to deal with the second order PDE

$$
\begin{aligned}
&(\{M_1^e\}+\frac{\delta t}{2}\{C_1^e\}+\frac{\delta t^2}{4}\{K_1^e\})\{DDX\}_{n+1}+\{D_1^e\}\{N\}_{n+1} \\
=\ &\{Q_1^e\}_{n+1}+\{S_1^e\}_{n+1}-\{C_1^e\}(\{DX\}_n+\frac{\delta}{2}\{DDX\}_n) \\
-\ &\{K_1^e\}(\{X\}_n+\delta t\{DX\}_n+\frac{\delta t^2}{4}\{DDX\}_n) \\
&\{X\}_{n+1}=\{X\}_n+\delta t\{DX\}_n+\frac{\delta t^2}{4}(\{DDX\}_n+\{DDX\}_{n+1}) \\
&\{DX\}_{n+1}=\{DX\}_n+\frac{\delta t}{2}(\{DDX\}_n+\{DDX\}_{n+1})
\end{aligned}
$$

## 4.3 Mathematical transformation of Second structural equation

$$
M_0\frac{\partial^2\eta}{\partial t^2}+L_r\frac{\partial\eta}{\partial t}+K_r\eta=(-\frac{E_\theta h}{a^2(1-\sigma_\theta\sigma_x)}+\frac{T_{\theta_0}}{a^2})\eta+T_{t_0}\frac{\partial^2\eta}{\partial x^2}
$$
$$
-\frac{E_\theta h\sigma_\theta}{a(1-\sigma_\theta\sigma_x)}\frac{\partial\xi}{\partial x}+[p-2\mu\frac{\partial u}{\partial r}]_a
$$

Strong form:

$$
\begin{aligned}
0 &= \int_{\Omega^e}\psi_i^e\{M_0\frac{\partial^2\eta}{\partial t^2}+L_r\frac{\partial\eta}{\partial t}+(K_r+\frac{E_\theta h}{a^2(1-\sigma_\theta\sigma_x)}-\frac{T_{\theta_0}}{a^2})\eta-T_{t_0}\frac{\partial^2\eta}{\partial x^2}\}dxdr \\
&+ \int_{\Omega^e}\psi_i^e\{\frac{E_\theta h\sigma_\theta}{a(1-\sigma_\theta\sigma_x)}\frac{\partial\xi}{\partial x}-[p-2\mu\frac{\partial u}{\partial r}]_a\}dxdr
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\Omega^e}\psi_i^e\{M_0\frac{\partial^2\eta}{\partial t^2}+L_r\frac{\partial\eta}{\partial t}+(K_r+\frac{E_\theta h}{a^2(1-\sigma_\theta\sigma_x)}-\frac{T_{\theta_0}}{a^2})\eta\}-T_{t_0}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\eta}{\partial x}dxdr \\
+\ &\int_{\Omega^e}\psi_i^e\frac{E_\theta h\sigma_\theta}{a(1-\sigma_\theta\sigma_x)}\frac{\partial\xi}{\partial x}dxdr=\int_{\Omega^e}\psi_i^e[p-2\mu\frac{\partial u}{\partial r}]_a dxdr+\oint_{\Gamma^e}\psi_i^e\frac{\partial\eta}{\partial x}n_x ds
\end{aligned}
$$

Semi-discretization:

$$
\eta(x,y)\approx\eta^e(x,y)=\sum_{j=1}^{n}N_j^e\psi_j^e(x,y)
$$

$$
\begin{aligned}
&\int_{\Omega^e}\psi_i^e\{M_0\frac{\partial^2\sum_{j=1}^{n}N_j^e\psi_j^e}{\partial t^2}+L_r\frac{\partial\sum_{j=1}^{n}N_j^e\psi_j^e}{\partial t} \\
+\ &(K_r+\frac{E_\theta h}{a^2(1-\sigma_\theta\sigma_x)}-\frac{T_{\theta_0}}{a^2})\sum_{j=1}^{n}N_j^e\psi_j^e\}dxdr \\
-\ &\int_{\Omega^e}\{T_{t_0}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\sum_{j=1}^{n}N_j^e\psi_j^e}{\partial x}+\psi_i^e\frac{E_\theta h\sigma_\theta}{a(1-\sigma_\theta\sigma_x)}\frac{\partial\sum_{j=1}^{n}X_j^e\psi_j^e}{\partial x}\}dxdr \\
=\ &\int_{\Omega^e}\psi_i^e[p-2\mu\frac{\partial u}{\partial r}]_a dxdr+\oint_{\Gamma^e}\psi_i^e\frac{\partial\eta}{\partial x}n_x ds
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{j=1}^{n}\int_{\Omega^e}\psi_i^e\psi_j^e\{M_0\frac{\partial^2 N_j^e}{\partial t^2}+L_r\frac{\partial N_j^e}{\partial t}\} \\
+ \quad & (\psi_i^e\psi_j^e(K_r+\frac{E_\theta h}{a^2(1-\sigma_\theta\sigma_x)})-\frac{T_{\theta_0}}{a^2}T_{t_0}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x})N_j^e\}dxdr \\
+ \quad & \sum_{j=1}^{n}\int_{\Omega^e}\{\psi_i^e\frac{E_\theta h\sigma_\theta}{a(1-\sigma_\theta\sigma_x)}\frac{\partial\psi_j^e}{\partial x}X_j^e\}dxdr \\
= \quad & \int_{\Omega^e}\psi_i^e[p-2\mu\frac{\partial u}{\partial r}]_a dxdr+\oint_{\Gamma^e}\psi_i^e\frac{\partial\eta}{\partial x}n_x ds
\end{aligned}
$$

### *4.3.1 1D version*

$$
\begin{aligned}
M_{ij}^e &= \int_{\Omega^e}M_0\psi_i^e\psi_j^e dxdr \\
&= 4\times DET_e[A200] \\
C_{ij}^e &= \int_{\Omega^e}L_r\psi_i^e\psi_j^e dxdr \\
&= 17\times 10^3\times DET_e[A200] \\
K_{ij}^e &= \int_{\Omega^e}(K_r\psi_i^e\psi_j^e+\frac{E_\theta h}{a^2(1-\sigma_\theta\sigma_x)}\psi_i^e\psi_j^e-\frac{T_{\theta_0}}{a^2}T_{t_0}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x})dxdr \\
&= (33\times 10^3+1.09\times 10^7)DET_e[A200] \\
D_{ij}^e &= \int_{\Omega^e}\{\psi_i^e\frac{E_\theta h\sigma_\theta}{a(1-\sigma_\theta\sigma_x)}\frac{\partial\psi_j^e}{\partial x}dxdr \\
&= 3.17\times 10^3\times DET_e J_1[A20x] \\
S_i^e &= \int_{\Omega^e}\psi_i^e[p-2\mu\frac{\partial u}{\partial r}]_a dxdr \\
&= [p-2\nu\frac{\partial u}{\partial r}]_a DET_e \\
Q_i^e &= \oint_{\Gamma^e}\psi_i^e\frac{\partial\eta}{\partial x}n_x ds = 0
\end{aligned}
$$

$$
\{M_2^e\}\frac{\partial^2}{\partial t^2}\{N^e\}+\{C_2^e\}\frac{\partial}{\partial t}\{N^e\}+\{K_2^e\}\{N^e\}+\{D_2^e\}\{X^e\} = \{Q_2^e\}+\{S_2^e\}
$$

### *4.3.2 Newmark Method*

Apply Newmark method to the structure equations to deal with the second order PDE

$$
\begin{aligned}
& (\{M_2^e\}+\frac{\delta t}{2}\{C_2^e\}+\frac{\delta t^2}{4}\{K_2^e\})\{DDN\}_{n+1}+\{D_2^e\}\{X\}_{n+1} \\
= \quad & \{Q_2^e\}_{n+1}+\{S_2^e\}_{n+1}-\{C_2^e\}(\{DN\}_n+\frac{\delta}{2}\{DDN\}_n) \\
- \quad & \{K_2^e\}(\{N\}_n+\delta t\{DN\}_n+\frac{\delta t^2}{4}\{DDN\}_n) \\
& \{N\}_{n+1}=\{N\}_n+\delta t\{DN\}_n+\frac{\delta t^2}{4}(\{DDN\}_n+\{DDN\}_{n+1}) \\
& \{DN\}_{n+1}=\{DN\}_n+\frac{\delta t}{2}(\{DDN\}_n+\{DDN\}_{n+1})
\end{aligned}
$$

## 4.4 2D version

2D version of structure equations are the same as the 1D version except for the boundary term and the use of elementary functions.

$$
\{M_1^e\}\frac{\partial^2}{\partial t^2}\{X^e\}+\{C_1^e\}\frac{\partial}{\partial t}\{X^e\}+\{K_1^e\}\{X^e\}+\{D_1^e\}\{N^e\}=\{Q_1^e\}+\{S_1^e\}
$$

$$
\begin{aligned}
M_{ij}^e &= \int_{\Omega^e}M_0\psi_i^e\psi_j^e dxdr \\
&= 4\times DET_e[Cn200] \\
C_{ij}^e &= \int_{\Omega^e}L_x\psi_i^e\psi_j^e dxdr \\
&= 17\times 10^3\times DET_e[Cn200] \\
K_{ij}^e &= \int_{\Omega^e}K_x\psi_i^e\psi_j^e-\frac{E_x h}{1-\sigma_\theta\sigma_x}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}dxdr \\
&= 33\times 10^3\times DET_e[Cn200]-13.1\times(J_1)^2 DET_e[Cn2xx] \\
D_i^e &= \int_{\Omega^e}\psi_i^e(\frac{E_x h\sigma_x}{a(1-\sigma_\theta\sigma_x)}+\frac{T_{t_0}-T_{\theta_0}}{a})\frac{\partial\psi_j^e}{\partial x}dxdr \\
&= 3.8\times 10^3\times DET_e J_1[Cn20x] \\
S_i^e &= \int_{\Omega^e}\psi_i^e\mu[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial x}]_a dxdr \\
&= 3.5\times 10^{-3}\times[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial x}]_a\times DET_e[Cn10] \\
Q_i^e &= \oint_{\Gamma^e}\psi_i^e q_n ds \\
&= q_n[A10]
\end{aligned}
$$

$$
\{M_2^e\}\frac{\partial^2}{\partial t^2}\{N^e\}+\{C_2^e\}\frac{\partial}{\partial t}\{N^e\}+\{K_2^e\}\{N^e\}+\{D_2^e\}\{X^e\}=\{Q_2^e\}+\{S_2^e\}
$$

$$
\begin{aligned}
M_{ij}^e &= \int_{\Omega^e}M_0\psi_i^e\psi_j^e dxdr \\
&= 4\times DET_e[Cn200] \\
C_{ij}^e &= \int_{\Omega^e}L_r\psi_i^e\psi_j^e dxdr \\
&= 17\times 10^3\times DET_e[Cn200] \\
K_{ij}^e &= \int_{\Omega^e}(K_r\psi_i^e\psi_j^e+\frac{E_\theta h}{a^2(1-\sigma_\theta\sigma_x)}\psi_i^e\psi_j^e-\frac{T_{\theta_0}}{a^2}T_{t_0}\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x})dxdr \\
&= (33\times 10^3+1.09\times 10^7)DET_e[Cn200] \\
D_{ij}^e &= \int_{\Omega^e}\{\psi_i^e\frac{E_\theta h\sigma_\theta}{a(1-\sigma_\theta\sigma_x)}\frac{\partial\psi_j^e}{\partial x}dxdr \\
&= 3.17\times 10^3\times DET_e J_1[Cn20x] \\
S_i^e &= \int_{\Omega^e}\psi_i^e[p-2\mu\frac{\partial u}{\partial r}]_a dxdr \\
&= [p-2\nu\frac{\partial u}{\partial r}]_a DET_e[Cn10] \\
Q_i^e &= \oint_{\Gamma^e}\psi_i^e\frac{\partial\eta}{\partial x}n_x ds=\frac{\partial\eta}{\partial x}n_x[A10]
\end{aligned}
$$

## 4.5 Combined System

$$\begin{aligned}
&(\{M_1^e\}+\frac{\delta t}{2}\{C_1^e\}+\frac{\delta t^2}{4}\{K_1^e\})\{DDX\}_{n+1}+\{D_1^e\}\{N\}_{n+1} && (4)\\
=\;& \{Q_1^e\}_{n+1}+\{S_1^e\}_{n+1}-\{C_1^e\}(\{DX\}_n+\frac{\delta}{2}\{DDX\}_n)\\
-\;& \{K_1^e\}(\{X\}_n+\delta t\{DX\}_n+\frac{\delta t^2}{4}\{DDX\}_n)
\end{aligned}$$

$$\begin{aligned}
&(\{M_2^e\}+\frac{\delta t}{2}\{C_2^e\}+\frac{\delta t^2}{4}\{K_2^e\})\{DDN\}_{n+1}+\{D_2^e\}\{X\}_{n+1} && (5)\\
=\;& \{Q_2^e\}_{n+1}+\{S_2^e\}_{n+1}-\{C_2^e\}(\{DN\}_n+\frac{\delta}{2}\{DDN\}_n)\\
-\;& \{K_2^e\}(\{N\}_n+\delta t\{DN\}_n+\frac{\delta t^2}{4}\{DDN\}_n)
\end{aligned}$$

$$\{X\}_{n+1}=\{X\}_n+\delta t\{DX\}_n+\frac{\delta t^2}{4}(\{DDX\}_n+\{DDX\}_{n+1}) \quad (6)$$

$$\{DX\}_{n+1}=\{DX\}_n+\frac{\delta t}{2}(\{DDX\}_n+\{DDX\}_{n+1}) \quad (7)$$

$$\{N\}_{n+1}=\{N\}_n+\delta t\{DN\}_n+\frac{\delta t^2}{4}(\{DDN\}_n+\{DDN\}_{n+1}) \quad (8)$$

$$\{DN\}_{n+1}=\{DN\}_n+\frac{\delta t}{2}(\{DDN\}_n+\{DDN\}_{n+1}) \quad (9)$$

Next, replace $\{N\}_{n+1} and \{X\}_{n+1}$ using (6),(8) and then change them into matrix form.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} DDX \\ DDN \end{pmatrix}=\begin{pmatrix} F1 \\ F2 \end{pmatrix}$$

$$\begin{aligned}
A &= \{M_1^e\}+\frac{\delta t}{2}\{C_1^e\}+\frac{\delta t^2}{4}\{K_1^e\}\\
B &= \frac{\delta t^2}{4}\{D_1^e\}\\
C &= \frac{\delta t^2}{4}\{D_2^e\}\\
D &= \{M_2^e\}+\frac{\delta t}{2}\{C_2^e\}+\frac{\delta t^2}{4}\{K_2^e\}\\
F1 &= \{Q_1^e\}_{n+1}+\{S_1^e\}_{n+1}-\{C_1^e\}(\{DX\}_n+\frac{\delta}{2}\{DDX\}_n)\\
&- \{K_1^e\}(\{X\}_n+\delta t\{DX\}_n+\frac{\delta t^2}{4}\{DDX\}_n)\\
&- \{D_1^e\}(\{N\}_n+\delta t\{DN\}_n+\frac{\delta t^2}{4}\{DDN\}_n)\\
F2 &= \{Q_2^e\}_{n+1}+\{S_2^e\}_{n+1}-\{C_2^e\}(\{DN\}_n+\frac{\delta}{2}\{DDN\}_n)\\
&- \{K_2^e\}(\{N\}_n+\delta t\{DN\}_n+\frac{\delta t^2}{4}\{DDN\}_n)\\
&- \{D_2^e\}(\{X\}_n+\delta t\{DX\}_n+\frac{\delta t^2}{4}\{DDX\}_n)
\end{aligned}$$

The 1D result and PICMSS code for 2D structure are present in the Appendix.

## 4.6 3D structural equation

D is the deformation matrix of vessel wall, and p is the pressure of the wall.

$$\begin{aligned}
\frac{\partial}{\partial x_1}(F_{11}^2+F_{12}^2-1-p)+\frac{\partial}{\partial x_2}(F_{21}F_{11}+F_{22}F_{12}) &= 0\\
\frac{\partial}{\partial x_1}(F_{21}F_{11}+F_{22}F_{12})+\frac{\partial}{\partial x_2}(F_{21}^2+F_{22}^2-1-p) &= 0\\
-\frac{\partial p}{\partial x_3} &= 0\\
F_{11}F_{22}-F_{21}F_{12} &= 0\\
F_{ij} &= \frac{\partial D_i}{\partial x_j}
\end{aligned}$$

# 5 Appendix

## 5.1 Formulation of Structure equations

The movement of the vessel wall can be described by balancing internal and external forces on a surface element of the vessel wall in its deformed state. It is convenient to change the variables to a coordinate system connected to the surface of the vessel. This is shown in the top part of Figure B.1. Let H be any vector pointing to the middle surface, as shown in Figure B.1:

$$H = x\hat{x}+R\hat{r}$$

where $\hat{x}$ and $\hat{r}$ are unit vectors in the cylindrical coordinate system in the longitudinal and radial directions, respectively, and $R(x,t)$ is the radius of the vessel. The new coordinates $(n,t,\theta)$ can be determined from $H$ . By assuming expressed in terms of $\hat{t}$ and $\hat{n}$ given by

$$\hat{t}=\frac{\frac{\partial H}{\partial x}}{|\frac{\partial H}{\partial x}|}=\frac{\hat{x}+\frac{\partial R}{\partial x}\hat{r}}{\sqrt{1+(\frac{\partial R}{\partial x})^2}} \quad and \quad \hat{n}=\frac{\hat{r}-\frac{\partial R}{\partial x}\hat{x}}{\sqrt{1+(\frac{\partial R}{\partial x}}} \quad (B.4)$$

because $\hat{t}$ and $\hat{n}$ are orthogonal. Solving for $\hat{x}$ and $\hat{r}$ gives

$$\hat{x}=\frac{\hat{t}-\frac{\partial R}{\partial x}\hat{n}}{\sqrt{1+(\frac{\partial R}{\partial x}}} \quad and \quad \hat{r}=\frac{\hat{n}+\frac{\partial R}{\partial x}\hat{t}}{\sqrt{1+(\frac{\partial R}{\partial x})^2}} \quad (B.5)$$

**Internal Forces**

The internal forces on the infinitesimal surface element $(dx \times rd\theta)$ have three components: a force $N$ across the vessel wall, a shearing force $S$ on the sides of the element, and a force $T$ normal to each of the edges; see the bottom part of Figure B.1. Most of these components are zero. The vessel wall is thin, and so any variation in the force across the wall can be neglected; i.e., $N_t = N_\theta = 0$. The flow is axisymmetric and without swirl. Hence no shearing force will act on the side of the element; i.e., $S_t = S_\theta = 0$. Thus the only forces left are $T_t$ and $T_\theta$ , the normal forces to each of the edges.

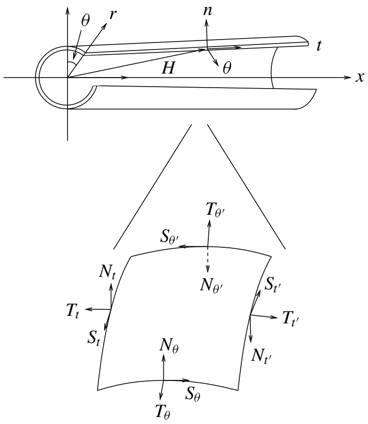



Figure B.1

**External Forces**

The internal forces must be balanced by external forces acting on the element. Let total external force be denoted by

$$P = P_t\hat{t} + P_n\hat{n} \tag{B.6}$$

where $P_t$ and $P_n$ are the tangential and normal components, respectively. $P$ can be split into inertial forces, tethering forces, and surface forces. In the following sections, these will be analyzed separately.

**Inertial Force**

Let $\xi(r,x,t)$ and $\eta(r,x,t)$ be the longitudinal and radial displacements of the wall. The inertial force per unit area is given by (see Atabek and Lew (1966))

$$T_{F_I} = -\rho_0 h(\frac{\partial^2\xi}{\partial t^2}\hat{x} + \frac{\partial^2\eta}{\partial t^2}\hat{r}), \tag{B.7}$$

where $\rho_0$ is the density and $h$ is the thickness of the wall. Because of the thin wall assumption, $h$ must be small compared to the vessel radius. We assume that both $\rho_0$ and $h$ are constant along any vessel of a given radius. The inertial force is the force ensuring that the internal and external forces are balanced. The inertial force must be included because the system is not steady, so it is necessary to take acceleration into account. In physics, this is known as d'Alambert's principle.

**Tethering Force**

The tethering force TFT can be modeled using a simple mechanical model consisting of a spring, a dash pot, and some lumped additional mass (Atabek, 1968). The tethering force (per unit area) acting in the radial and longitudinal dir(ections is given by

$$T_{F_T} = -(M_a\frac{\partial^2\xi}{\partial t^2} + L_x\frac{\partial\xi}{\partial t} + K_x\xi)\hat{x} - (M_a\frac{\partial^2\eta}{\partial t^2} + L_r\frac{\partial\eta}{\partial t} + K_r\eta)\hat{r}, \tag{B.8}$$

where $K_i$ and $L_i$ ,$i = x, r$,are the spring and frictional coefficients of the dash pot in the $i$th direction and $M_a$ is the additional mass of the system. These are assumed to be the same in both directions. Since both inertial and tethering forces act in the same direction, it is convenient to add them before projecting the forces in the normal and tangential directions. Let

$$M_0 = M_a + \rho_0 h.$$

The resultant inertial and tethering force in the tangential and normal directions, respectively, then yield

$$T_{F_{Tres}} \cdot \hat{t} \tag{B.9}$$

$$= -\left[\left(M_0\frac{\partial^2\xi}{\partial t} + L_x\frac{\partial\xi}{\partial t} + K_x\xi\right) + \left(M_0\frac{\partial^2\eta}{\partial t^2} + L_r\frac{\partial\eta}{\partial t} + K_r\eta\right)\frac{\partial R}{\partial x}\right] \Bigg/ \sqrt{1+\left(\frac{\partial R}{\partial x}\right)}$$

$$T_{F_{Tres}} \cdot \hat{n} \tag{B.10}$$

$$= \left[\left(M_0\frac{\partial^2\xi}{\partial t} + L_x\frac{\partial\xi}{\partial t} + K_x\xi\right)\frac{\partial R}{\partial x} - \left(M_0\frac{\partial^2\eta}{\partial t^2} + L_r\frac{\partial\eta}{\partial t} + K_r\eta\right)\right] \Bigg/ \sqrt{1+\left(\frac{\partial R}{\partial x}\right)^2}$$

**Surface Force**

The surface force is a result of fluid interaction with the vessel wall. If the stress tensor of the fluid is given by $T_{F_S}$ , then interaction with the inner vessel wall (at $r = R?h/2 = a$) is given by $-T_{F_S}\cdot\hat{n}$. Assume that the stress tensor can be separated into radial and longitudinal directions

$$(-T_{F_S}\cdot\hat{n})\cdot\hat{t} \quad \textit{and} \quad (-T_{F_S}\cdot\hat{n})\cdot\hat{n} \tag{B.11}$$

The stress tensor for incompressible flow is given by Ockendon and Ockendon (1995):

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

In cylindrical coordinates the stress tensor becomes

$$T_{F_S} = \begin{bmatrix} T_{rr} & T_{rx} \\ T_{rx} & T_{xx} \end{bmatrix}_a = \begin{bmatrix} -p+2\mu\frac{\partial u}{\partial r} & \mu(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial x}) \\ \mu(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial x}) & -p+2\mu\frac{\partial w}{\partial x} \end{bmatrix}_a \tag{B.12}$$

The fluid stress in the $\hat{t}$ and $\hat{n}$ directions can be found as

$$(-T_{F_S}\cdot\hat{n})\cdot\hat{t} = \left[(T_{xx}-T_{rr})\frac{\partial R}{\partial x} + T_{rx}\left(\left(\frac{\partial R}{\partial x}\right)^2 - 1\right)\right]_a \Bigg/ \left(1+\left(\frac{\partial R}{\partial x}\right)^2\right). \tag{B.13}$$

$$(-T_{F_S}\cdot\hat{n})\cdot\hat{n} = \left[2T_{rx}\frac{\partial R}{\partial x} - T_{rr} - T_{xx}\left(\frac{\partial R}{\partial x}\right)^2\right]_a \Bigg/ \left(1+\left(\frac{\partial R}{\partial x}\right)^2\right). \tag{B.14}$$

**Total External Force**

The total external force can be found by adding the inertial and tethering forces (B.9) and (B.10) as well as the surface forces (B.13) and (B.14). Generally, these forces are not estimated at the same point, but because of the thin wall assumption the resulting error in the total external force is negligible. Equation (B.6) gives

$$P = P_t\hat{t} + P_n\hat{n} = (-T_{F_S}\cdot\hat{n} + T_{F_{Tres}})\cdot\hat{t} + (-T_{F_S}\cdot\hat{n} + T_{F_{Tres}})\cdot\hat{n}.$$

The tangential component is

$$P_t = \left[(T_{xx}-T_{rr})\frac{\partial R}{\partial x} + T_{rx}\left(\left(\frac{\partial R}{\partial x}\right)^2 - 1\right)\right]_a \Bigg/ \left(1+\left(\frac{\partial R}{\partial x}\right)^2\right) \tag{B.15}$$

$$-\left(\left(M_0\frac{\partial^2\xi}{\partial t} + L_x\frac{\partial\xi}{\partial t} + K_x\xi\right) + \left(M_0\frac{\partial^2\eta}{\partial t^2} + L_r\frac{\partial\eta}{\partial t} + K_r\eta\right)\frac{\partial R}{\partial x}\right) \Bigg/ \sqrt{1+\left(\frac{\partial R}{\partial x}\right)^2}$$

and the normal component is

$$P_n = \left[2T_{rx}\frac{\partial R}{\partial x} - T_{rr} - T_{xx}\left(\frac{\partial R}{\partial x}\right)^2\right]_a \Bigg/ \left(1+\left(\frac{\partial R}{\partial x}\right)^2\right) \tag{B.16}$$

$$+\left(\left(M_0\frac{\partial^2\xi}{\partial t} + L_x\frac{\partial\xi}{\partial t} + K_x\xi\right)\frac{\partial R}{\partial x} - \left(M_0\frac{\partial^2\eta}{\partial t^2} + L_r\frac{\partial\eta}{\partial t} + K_r\eta\right)\right) \Bigg/ \sqrt{1+\left(\frac{\partial R}{\partial x}\right)^2}$$

**Balancing Internal and External Forces** When a wave is propagated along a vessel, the vessel will dilate. Hence the surface will appear as shown in Figure B.2. Considering this surface, we can derive the equilibrium equations. Balancing of internal and external forces will also be carried out in two parts: one for tangential contributions and one for normal contributions.

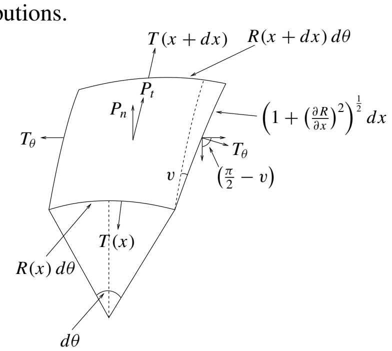



Figure B.2

**Balancing Tangential Components of Internal and External Forces** The area of the surface in Figure B.2 is given by $Rd\theta\sqrt{1+(\partial R/\partial x)^2}dx$, and the tangential part $P_{tan}$ of the external strain $P_t$ is given by

$$P_{tan} = P_t R d\theta \sqrt{1+\left(\frac{\partial R}{\partial x}\right)^2}dx.$$

The pressure load on any given volume element is $-P_{ext}$ . This should be balanced by the internal stress over the surface element projected in the tangential direction. Thus the stress over the surface in the tangential direction is given by

$$T_{tan_1} = -T_t(x)R(x)d\theta + T_t(x+dx)R(x+dx)d\theta \approx \frac{\partial}{\partial x}(T_t R)dxd\theta,$$

where the last equality is approximated using the first order Taylor expansion for $T_t t(x+dx)R(x+dx)$.
Furthermore, the stress from the radial tension also contributes. As seen on the right- hand side of the surface element in Figure B.2, the radial tension $T_\theta$ gives contributions in both the tangential and the radial directions. Since we have axial symmetry, the net tension around the vessel at any location is zero. The part of $T_\theta$ pointing backward in the tangential direction is given by

$$T_{tan_2} = -T_\theta\cos\left(\frac{\pi}{2}-v\right)\sqrt{1+\left(\frac{\partial R}{\partial x}\right)^2}dx = -T_\theta\frac{\partial R}{\partial x}d\theta dx,$$

where v is defined as shown in Figure B.2. Balancing $T_{tan_1}$ and $T_{tan_2}$ with $P_{tan}$ and dividing by $d\theta dx$ gives

$$-T_\theta\frac{\partial R}{\partial x} + \frac{\partial}{\partial x}(RT_t) + P_t R\sqrt{1+\left(\frac{\partial R}{\partial x}\right)^2} = 0. \tag{B.17}$$

**Balancing Normal Components of Internal and External Forces** Balancing normal internal stresses with the normal external strain gives

$$P_n = \kappa_\theta T_\theta + \kappa_t T_t,$$

where $\kappa_i, i=\theta, t$ , is the curvature in the $i$ direction. As seen in Figure B.3, the curvatures in the longitudinal and angular directions are given by

$$\kappa_\theta = \frac{1}{R}\Bigg/\sqrt{1+\left(\frac{\partial R}{\partial x}\right)^2} \quad and \quad \kappa_t = -\frac{\partial^2 R}{\partial x^2}\Bigg/\sqrt{1+\left(\frac{\partial R}{\partial x}\right)^2}^3.$$

Hence the balancing equation becomes

$$\kappa_\theta T_\theta + \kappa_t T_t - P_n = 0$$

$$\Leftrightarrow \frac{T_\theta}{R} - T_t\frac{\partial^2 R}{\partial x^2}\Bigg/\left(1+\left(\frac{\partial R}{\partial x}\right)^2\right) - P_n\sqrt{1+\left(\frac{\partial R}{\partial x}\right)} = 0. \tag{B.18}$$

Inserting (B.15) and (B.16) into (B.17) and (B.18) gives

$$T_\theta\frac{\partial R}{\partial x} + \frac{\partial}{\partial x}(RT_t)$$

$$-R\left(M_0\frac{\partial^2\xi}{\partial t} + L_x\frac{\partial\xi}{\partial t} + K_x\xi + \left(M_0\frac{\partial^2\eta}{\partial t^2} + L_r\frac{\partial\eta}{\partial t} + K_r\eta\right)\frac{\partial R}{\partial x}\right)$$

$$+R\left[(T_{xx}-T_{rr})\frac{\partial R}{\partial x} + T_{rx}\left(\left(\frac{\partial R}{\partial x}\right)^2 - 1\right)\right]_a \Bigg/ \sqrt{1+\left(\frac{\partial R}{\partial x}\right)^2} = 0, \tag{B.19}$$

$$\frac{T_\theta}{R} - T_t\frac{\partial^2 R}{\partial x^2}\Bigg/\left(1+\left(\frac{\partial R}{\partial x}\right)^2\right)$$

$$-\left(M_0\frac{\partial^2\xi}{\partial t} + L_x\frac{\partial\xi}{\partial t} + K_x\xi\right)\frac{\partial R}{\partial x} + M_0\frac{\partial^2\eta}{\partial t^2} + L_r\frac{\partial\eta}{\partial t} + K_r\eta$$

$$\left[2T_{rx}\frac{\partial R}{\partial x} - T_{rr} - T_{xx}\left(\frac{\partial R}{\partial x}\right)^2\right]_a \Bigg/ \sqrt{1+\left(\frac{\partial R}{\partial x}\right)^2} = 0. \tag{B.20}$$

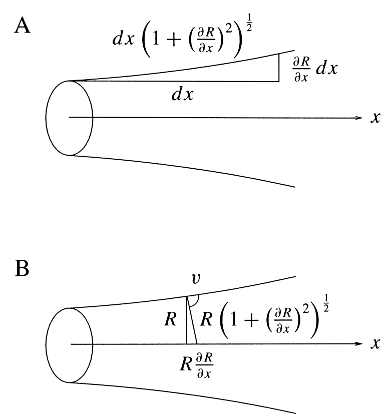



**Figure B.3.** *Curvature of the vessel. The longitudinal curvature (in* A*) is given by* $\kappa_t$*, and the tangential curvature normal to the surface (in* B*) is given by* $\kappa_\theta$*.*

**Elasticity Relations** The purpose of this section is to set up stress-strain relations such that the stress components $T_i$can be related to the displacements of the wall $(\xi,\eta)$. These are measured from some reference state where vessels are stretched to their in vivo length. The reason is that a loose piece of artery (unstressed) requires very large deformations to be brought to its original stressed state. However, the general theory of elasticity applies only for small deformations; see,e.g.,LandauandLifshitz(1986). Thisproblemcanbeavoid-edbymakingthederivations orginate from some initial stressed state. Hence it is assumed that, when a wave moves along an artery, it undergoes small deformations from its reference state. The initial state is chosen to be the state where the transmural pressure of the artery is zero. Furthermore, it is assumed that it is adequate to apply a linear relation between stress and strain. Let the reference state of stresses in the longitudinal and circumferential directions be denoted by $T_{t_0}$ and $T_{\theta_0}$ . Then the following relations can be obtained:

$$T_\theta - T_{\theta_0} = \frac{E_\theta h}{1-\sigma_\theta\sigma_x}(\varepsilon_r + \sigma_x\varepsilon_x) \quad and \quad T_t - T_{t_0} = \frac{E_x h}{1-\sigma_\theta\sigma_x}(\varepsilon_x + \sigma_\theta\varepsilon_r), \tag{B.21}$$

where $E_i$, $i=\theta,t$, is Young's modulus in the $i$th direction; $h$ is the wall thickness; $\sigma_i$, $i=\theta,x$,is the Poisson ratio in the $i$th direction; and $\varepsilon_i$,$i=\theta,x$,is the displacement relative to the reference state; see, e.g., Landau and Lifshitz (1986). The relative circumferential and longitudinal displacements are given by

$$\varepsilon_r = \frac{\eta}{R} \quad and \quad \varepsilon_x = \frac{\partial\xi}{\partial x}$$

**Balancing Fluid and Wall Motions** Boundary conditions linking the velocity of the wall to the velocity of the fluid remain to be specified. Assume that the fluid particles are at rest at the wall. Hence

$$[u]_{r=a} = \frac{\partial\eta}{\partial t} \quad and \quad [w]_{r=a} = \frac{\partial\xi}{\partial t} \tag{B.22}$$

Furthermore, assume that the component of the fluid velocity normal to the wall is equal to the normal velocity of the inner surface of the vessel wall. Hence the normal velocity of the wall, at $a = R(x+\xi,t) - h/2$, is given by

$$\frac{d}{dt}\left(r - R + \frac{h}{2}\right) = 0 \Leftrightarrow [u]_{r=a} - [w]_{r=a}\frac{\partial R}{\partial x} - \frac{\partial R}{\partial t} = 0$$

**Linearization** In principle the correct number of equations and boundary conditions are present. However, in their present form these equations are too complicated to solve analytically. As discussed earlier, the purpose was to set up a simple system of equations for the smaller arteries. Therefore, following Atabek and Lew (1966), we have chosen to linearize them. The linearization is based on expansion of the dependent variables in power series of a small parameter $\varepsilon$ around a known solution. This is defined by a situation where the fluid is at rest and the vessel is inflated and stretched. Furthermore, if $\varepsilon = 0$, then all dependent variables give the known solution. The expansion is given by

$$s = s_1\varepsilon + s_2\varepsilon^2 + \cdots \quad for\ s = u, w, \eta, \xi, T_{rx}, \tag{B.23}$$

$$\tilde{s} = \tilde{s}_0 + \tilde{s}_1\varepsilon + \tilde{s}_2\varepsilon^2 + \cdots \quad for\ \tilde{s} = p, R, T_\theta, T_t, T_{rr}, T_{xx}, \tag{B.24}$$

where $s_0$ is a constant defining the reference state (at zero transmural pressure). Let $f(r,x,t)$ be either of the functions in (B.23) or (B.24). In order to accomplish the linearization,$f(r,x,t)$ must be evaluated at $r = a = R - h/2$. The power series expansion together with the Taylor series expansion to first order yields

$$\begin{aligned} f(r,x,t) &\approx f(a,x,t) + f'(r,x,t)(r-a) \\ &= f_0(a,x,t) + f_1(a,x,t)\varepsilon \\ &\quad + (f_0'(a,x,t) + f_1'(a,x,t)\varepsilon)(r - (R_0 + R_1\varepsilon h/2)) \\ &= f_0(a,x,t) + kf_0'(a,x,t) \\ &\quad + \varepsilon(f_1(a,x,t) - R_1 f_0'(a,x,t) + kf_1'(a,x,t)), \end{aligned} \tag{B.25}$$

where $k = r - R_0 + h/2$. Using (B.23) to (B.25), the zeroth and first order equations can be obtained by assembling terms to the respective powers of $\varepsilon$ from the nonlinear equations (B.1) to (B.3), (B.19), and (B.20). **Terms of First Order Approximations** The first order terms of the shell equation (B.19) give

$$-T_{\theta_0}\frac{\partial R_1}{\partial x} + \frac{\partial}{\partial x}(R_0 T_{t_1} + R_1 T_{t_0}) \tag{B.30}$$

$$-R_0\left(M_0\frac{\partial^2\xi_1}{\partial t^2} + L_x\frac{\partial\xi_1}{\partial t} + K_x\xi_1 - \left[(T_{xx_0} - T_{rr_0})\frac{\partial R_1}{\partial x} - T_{rx_1}\right]_a\right) = 0$$

$$\Leftrightarrow M_0\frac{\partial^2\xi_1}{\partial t^2} + L_x\frac{\partial\xi_1}{\partial t} + K_x\xi_1 = \frac{\partial T_{t_1}}{\partial x} + \frac{T_{t_0} - T_{\theta_0}}{R_0}\frac{\partial R_1}{\partial x} - \mu\left[\frac{\partial w_1}{\partial r} + \frac{\partial u_1}{\partial x}\right]_a.$$

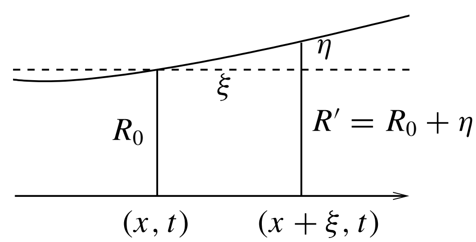



**Figure B.4.** *Estimation of $R(x + \xi, t)$ using the definitions of $\xi$ and $\eta$.*

The last equation is obtained using the stress tensor (B.12) for the first order approximation of $T_{rx_1}$ and the zeroth order approximation of $T_{xx_0} - T_{rr_0}$ , which cancel. The first order terms of the shell equation (B.20) give

$$\frac{T_{\theta_1}}{R_0} - T_{\theta_0}\frac{R_1}{R_0^2} - T_{t_0}\frac{\partial^2 R_1}{\partial x^2} - M_0\frac{\partial^2\eta_1}{\partial t^2} + L_r\frac{\partial\eta_1}{\partial t} + K_r\eta_1 + T_{rr_1} = 0$$

$$\Leftrightarrow M_0\frac{\partial^2\eta_1}{\partial t^2} + L_r\frac{\partial\eta_1}{\partial t} + K_r\eta_1 = -\frac{T_{\theta_1}}{R_0} + T_{\theta_0}\frac{R_1}{R_0^2} + T_{t_0}\frac{\partial^2 R_1}{\partial x^2} + \left[p_1 - 2\mu\frac{\partial u_1}{\partial r}\right]_a.$$

where we have again used (B.12) for the first order approximation $T_{rr_1}$ . Assuming that the second order approximations can be neglected, $\varepsilon$ can be incorporated into the dependent variables and we can set $\varepsilon = 1$. For any $(x,t)$ the first order Taylor expansion of $R(x+\xi,t)$ gives

$$R(x+\xi,t) = R(x,t) + \frac{\partial R}{\partial x}\xi = R_0 + \eta,$$

as seen in Figure B.4. The first order expansion of $R$ from (B.24) is given by

$$R(x,t) = R_0 + R_1\varepsilon + O(\varepsilon^2) = R_0 + \eta_1\varepsilon + O(\varepsilon^2) \Leftrightarrow \eta_1 = R_1,$$

since $\eta$ has no zeroth order term. Furthermore, we approximate $R_0$ by the inner radius $a = R_0 - h/2$. Since the walls are assumed to be thin compared with the vessel radius, i.e., $h \ll a$, the error is negligible. Finally, the indices 1 are dropped and the definitions in (B.21) are used for $T_{\theta_1}$ and $T_{t_1}$ . The linearized equations can be obtained from their first order approximations; i.e., (B.30) and (B.31) become

$$M_0 \frac{\partial^2 \xi}{\partial t^2} + L_x \frac{\partial \xi}{\partial t} + K_x \xi$$

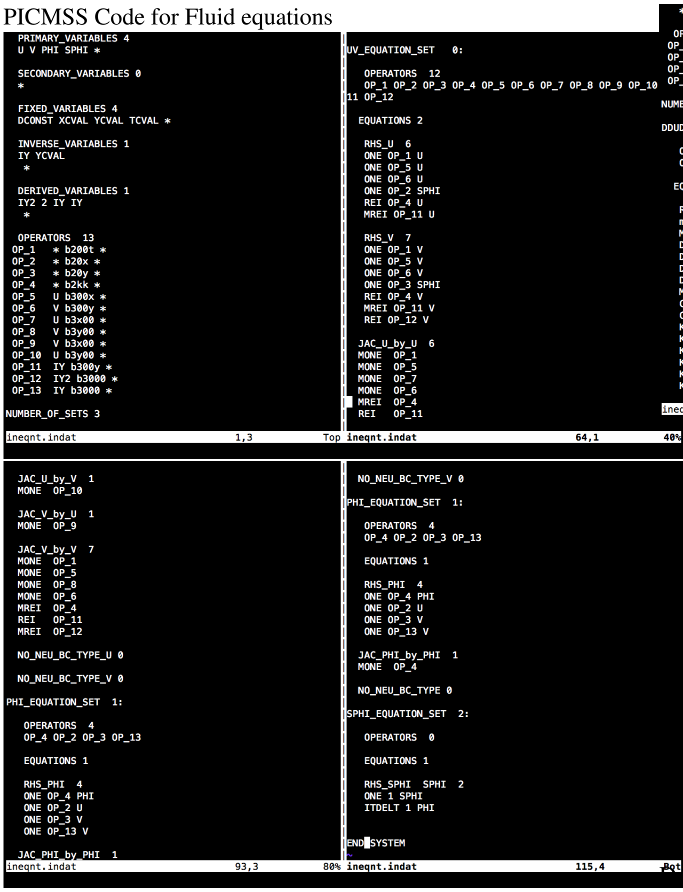
$$= \frac{E_x h}{1-\sigma_\theta \sigma_x} \frac{\partial^2 \xi}{\partial x^2} + \left(\frac{E_x h \sigma_x}{a(1-\sigma_\theta \sigma_x)} + \frac{T_{t_0} - T_{\theta_0}}{a}\right) \frac{\partial \eta}{\partial x} - \mu\left[\frac{\partial w}{\partial r} + \frac{\partial u}{\partial x}\right]_a, \tag{B.32}$$

$$M_0 \frac{\partial^2 \eta}{\partial t^2} + L_r \frac{\partial \eta}{\partial t} + K_r \eta$$

$$= \left(-\frac{E_\theta h}{a^2(1-\sigma_\theta \sigma_x)} + \frac{T_{\theta_0}}{a^2}\right)\eta + T_{t_0} \frac{\partial^2 \eta}{\partial x^2} - \frac{E_\theta h \sigma_\theta}{a(1-\sigma_\theta \sigma_x)} \frac{\partial \xi}{\partial x} + [p - 2\mu \frac{\partial u}{\partial r} \tag{B.33}$$

## 5.2 Result

PICMSS Code for Fluid equations



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  FIXED_VARIABLES 4
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  IY YCVAL
   *

  DERIVED_VARIABLES 1
  IY2 2 IY IY
   *

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 OP_2   * b20x *
 OP_3   * b20y *
 OP_4   * b2kk *
 OP_5   U b300x *
 OP_6   V b300y *
 OP_7   U b3x00 *
 OP_8   V b3y00 *
 OP_9   V b3x00 *
 OP_10  U b3y00 *
 OP_11  IY b300y *
 OP_12  IY2 b3000 *
 OP_13  IY b3000 *

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11 OP_12

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   ONE OP_5 U
   ONE OP_6 U
   ONE OP_2 SPHI
   REI OP_4 U
   MREI OP_11 U

   RHS_V  7
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   ONE OP_5 V
   ONE OP_6 V
   ONE OP_3 SPHI
   REI OP_4 V
   MREI OP_11 V
   REI OP_12 V

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  MONE  OP_7
  MONE  OP_6
  MREI  OP_4
  REI   OP_11
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  MONE  OP_9

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  MONE  OP_6
  MREI  OP_4
  REI   OP_11
  MREI  OP_12

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   ONE OP_3 V
   ONE OP_13 V

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   ONE OP_3 V
   ONE OP_13 V

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   ITDELT 1 PHI


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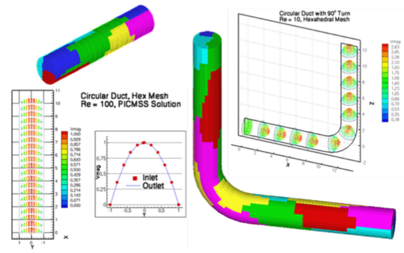
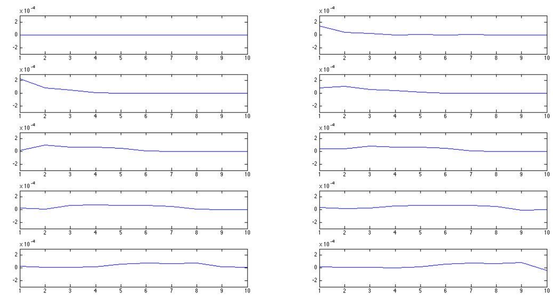
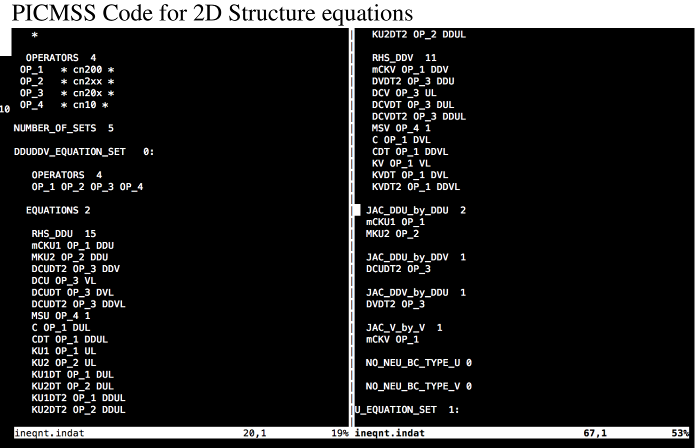
Result for Fluid equations




1D Structure equations' result



PICMSS Code for 2D Structure equations



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   DCUDT2 OP_3 DDVL
   MSU OP_4 1
   C OP_1 DUL
   CDT OP_1 DDUL
   KU1 OP_1 UL
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